# Principles of Robot Autonomy II

Markov decision processes and dynamic programming

Stanford University

Autonomous Systems Lab

# Today's lecture

- Aim
  - Learn the fundamental principles of Markov decision processes and dynamic programming

- Readings
  - D. Bertsekas. Reinforcement Learning and Optimal Control, 2019. Chapters 1 and 2.

# Basic decision-making problem (deterministic)

- System: $\mathbf{x}_{k+1} = f_k(\mathbf{x}_k, \mathbf{u}_k), \quad k = 0, \dots, N$
- Control constraints: $\mathbf{u}_k \in U(\mathbf{x}_k)$
- Cost:

$$J(\mathbf{x}_0; \boldsymbol{u}_0, \dots, \boldsymbol{u}_{N-1}) = g_N(\mathbf{x}_N) + \sum_{k=0}^{N-1} g_k(\mathbf{x}_k, \mathbf{u}_k)$$

- Decision-making problem:

$$J^*(\mathbf{x}_0) = \min_{\mathbf{u}_k \in U(\mathbf{x}_k),\, k=0,\dots,N-1} J(\mathbf{x}_0; \boldsymbol{u}_0, \dots, \boldsymbol{u}_{N-1})$$

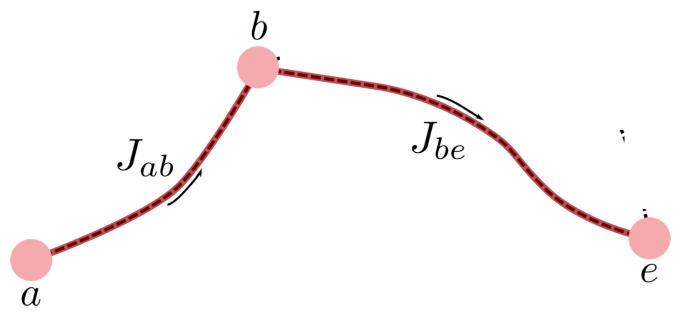
# Key points

- Discrete-time model
- Additive cost (central assumption)

# Principle of optimality

The key concept behind the dynamic programming approach is the principle of optimality

Suppose optimal path for a multi-stage decision-making problem is




- first decision yields segment $a - b$ with cost $J_{ab}$
- remaining decisions yield segments $b - e$ with cost $J_{be}$
- optimal cost is then $J_{ae}^* = J_{ab} + J_{be}$

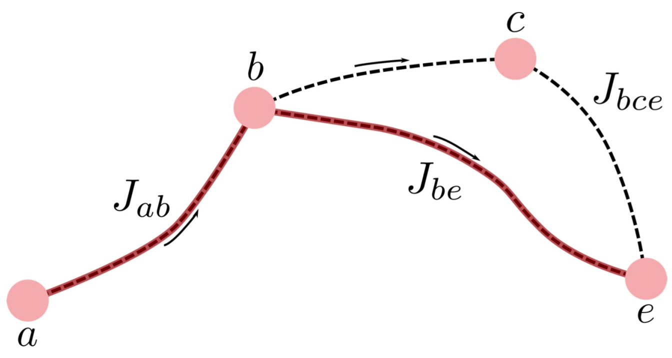
# Principle of optimality

- Claim: If $a - b - e$ is optimal path from $a$ to $e$, then $b - e$ is optimal path from $b$ to $e$
- *Proof*: Suppose $b - c - e$ is the optimal path from $b$ to $e$. Then

$$J_{bce} < J_{be}$$

and

$$J_{ab} + J_{bce} < J_{ab} + J_{be} = J_{ae}^{*}$$




Contradiction!

# Principle of optimality

Principle of optimality (for deterministic systems): Let $\{\mathbf{u}_0^*, \mathbf{u}_1^*, \dots, \mathbf{u}_{N-1}^*\}$ be an optimal control sequence, which together with $\mathbf{x}_0^*$ determines the corresponding state sequence $\{\mathbf{x}_0^*, \mathbf{x}_1^*, \dots, \mathbf{x}_N^*\}$. Consider the subproblem whereby we are at $\mathbf{x}_k^*$ at time $k$ and we wish to minimize the cost-to-go from time $k$ to time $N$, i.e.,

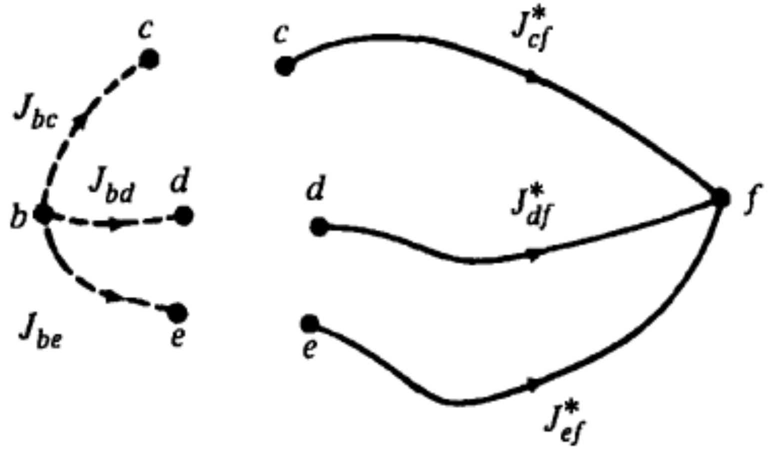
$$g_k(\mathbf{x}_k^*, \mathbf{u}_k) + \sum_{m=k+1}^{N-1} g_m(\mathbf{x}_m, \mathbf{u}_m) + g_N(\mathbf{x}_N)$$

Then the truncated optimal sequence $\{\mathbf{u}_k^*, \mathbf{u}_{k+1}^*, \dots, \mathbf{u}_{N-1}^*\}$ is optimal for the subproblem

- Tail of optimal sequences optimal for tail subproblems

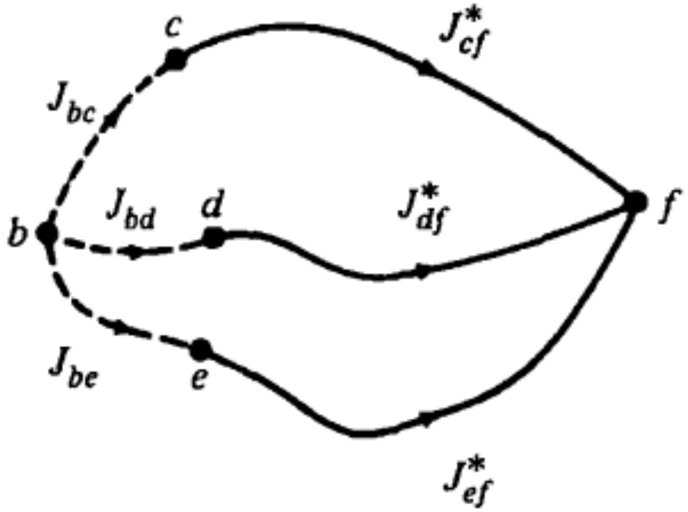
# Applying the principle of optimality

Principle of optimality: if $b - c$ is the initial segment of the optimal path from $b$ to $f$, then $c - f$ is the terminal segment of this path




Hence, the optimal trajectory is found by comparing:

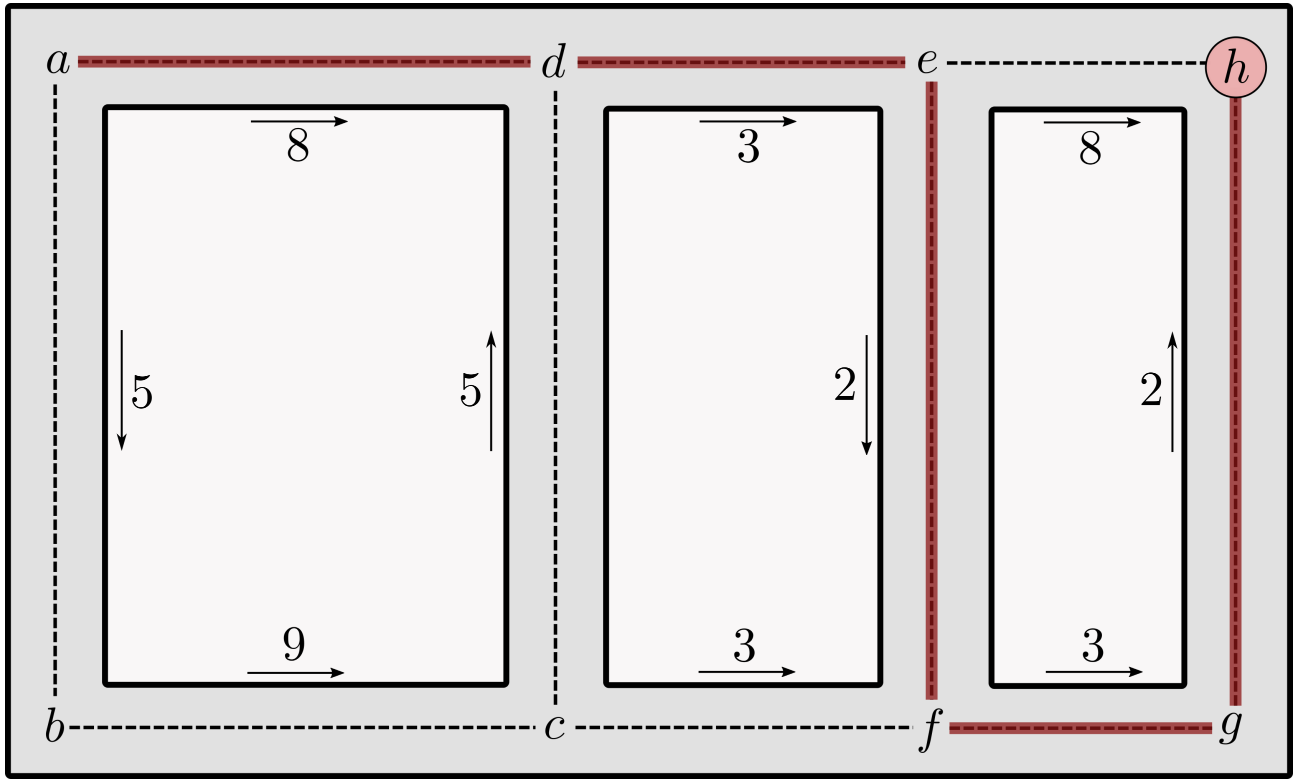
$$C_{bcf} = J_{bc} + J^*_{cf}$$
$$C_{bdf} = J_{bd} + J^*_{df}$$
$$C_{bef} = J_{be} + J^*_{ef}$$



# Applying the principle of optimality

- need only to compare the concatenations of immediate decisions and optimal decisions → significant decrease in computation / possibilities
- in practice: carry out this procedure backward in time

# Example




Optimal cost: 18

Optimal path: $a \rightarrow\ d \rightarrow\ e \rightarrow\ f \rightarrow\ g \rightarrow\ h$

# DP Algorithm

- Start with

$$J_N^*(\mathbf{x}_N) \; = \; g_N(\mathbf{x}_N), \;\; \text{for all } \mathbf{x}_N$$

- and for $k = N - 1, \dots, 0$, let

$$J_k^*(\mathbf{x}_k) = \min_{\mathbf{u}_k \in U(\mathbf{x}_k)} g(\mathbf{x}_k, \mathbf{u}_k) + J_{k+1}^*\big(f(\mathbf{x}_k, \mathbf{u}_k)\big) \quad \text{for all } \mathbf{x}_k$$

Once the functions $J_0^*, \dots, J_N^*$ have been determined, the optimal sequence can be determined with a forward pass

# Comments

- discretization (from differential equations to difference equations)
- quantization (from continuous to discrete state variables / controls)
- global minimum
- constraints, in general, simplify the numerical procedure
- curse of dimensionality

# Basic decision-making problem (stochastic)

- System: $\mathbf{x}_{k+1} = f_k(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k), \quad k = 0, \dots, N-1$
- Control constraints: $\mathbf{u}_k \in U(\mathbf{x}_k)$
- Probability distribution: $P_k(\cdot \mid \mathbf{x}_k, \mathbf{u}_k)$
- Policies: $\pi = \{\pi_0, \dots, \pi_{N-1}\}$, where $\mathbf{u}_k = \pi_k(\mathbf{x}_k)$
- Expected cost:

$$J_\pi(\mathbf{x}_0) = E_{\mathbf{w}_k, k=0,\dots,N-1}\left\{g_N(\mathbf{x}_N) + \sum_{k=0}^{N-1} g_k(\mathbf{x}_k, \pi_k(\mathbf{x}_k), \mathbf{w}_k)\right\}$$

- Decision-making problem:

$$J^*(\mathbf{x}_0) = \min_\pi \; J_\pi(\mathbf{x}_0)$$

# Key points

- Discrete-time model
- Markovian model
- Objective: find optimal closed-loop policy
- Additive cost (central assumption)
- Risk-neutral formulation

Other communities use different notation:

- Powell, W. B. AI, OR and control theory: A Rosetta Stone for stochastic optimization. Princeton University, 2012. http://castlelab.princeton.edu/Papers/AIOR_July2012.pdf

# Principle of optimality

Principle of optimality (for stochastic systems): Let $\pi^* := \{\pi_0^*, \pi_1^*, \dots, \pi_{N-1}^*\}$ be an optimal policy. Assume state $\mathbf{x}_k$ is reachable. Consider the subproblem whereby we are at $\mathbf{x}_k$ at time $k$ and we wish to minimize the cost-to-go from time $k$ to time $N$. Then the truncated policy $\{\pi_k^*, \pi_{k+1}^*, \dots, \pi_{N-1}^*\}$ is optimal for the subproblem

- tail policies optimal for tail subproblems

# DP Algorithm

DP Algorithm: For every initial state $\mathbf{x}_0$, the optimal cost $J^*(\mathbf{x}_0)$ is equal to $J_0(\mathbf{x}_0)$, given by the last step of the following algorithm, which proceeds backward in time from stage $N-1$ to stage 0:

$$J_N(\mathbf{x}_N) \;=\; g_N(\mathbf{x}_N)$$

$$J_k(\mathbf{x}_k) = \min_{\mathbf{u}_k \in U(\mathbf{x}_k)} E_{\mathbf{w}_k}\{g_k(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k) + J_{k+1}(f_k(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k))\}, \quad k = 0, \ldots, N-1$$

Furthermore, if $\mathbf{u}_k^* = \pi_k^*(\mathbf{x}_k)$ minimizes the right-hand side of the above equation for each $\mathbf{x}_k$ and $k$, the policy $\{\pi_0^*, \pi_1^*, \ldots, \pi_{N-1}^*\}$ is optimal

# Example: Inventory Control Problem (1/3)

- Stock available $x_k \in \mathbb{N}$, inventory $u_k \in \mathbb{N}$, and demand $w_k \in \mathbb{N}$
- Dynamics: $x_{k+1} = \max(0, x_k + u_k - w_k)$
- Constraints: $x_k + u_k \leq 2$
- Probabilistic structure: $p(w_k = 0) = 0.1$, $p(w_k = 1) = 0.7$, and $p(w_k = 2) = 0.2$
- Cost

$$E\left\{ \underbrace{0}_{g_3(x_3)} + \sum_{k=0}^{2} \underbrace{\left( u_k + (x_k + u_k - w_k)^2 \right)}_{g_k(x_k, u_k, w_k)} \right\}$$

# Example: Inventory Control Problem (2/3)

- Algorithm takes form for $k = 0,1,2$

$$J_k(x_k) = \min_{0 \leq u_k \leq 2 - x_k} E_{w_k} \left\{ u_k + (x_k + u_k - w_k)^2 + J_{k+1}\left(\max(0, x_k + u_k - w_k)\right) \right\}$$

- For example

$$J_2(0) = \min_{u_2 = 0,1,2} E_{w_2} \{u_2 + (u_2 - w_2)^2\} =$$

$$\min_{u_2 = 0,1,2} \{u_2 + 0.1(u_2)^2 + 0.7(u_2 - 1)^2 + 0.2(u_2 - 2)^2\}$$

which yields $J_2(0) = 1.3$, and $\pi_2^*(0) = 1$

# Example: Inventory Control Problem (3/3)

Final solution:

- $J_0(0) \ = \ 3.7,$
- $J_0(1) \ = \ 2.7,$ and
- $J_0(2) \ = \ 2.818$

(see this spreadsheet)

# Difficulties of DP

- Curse of dimensionality:
  - Exponential growth of the computational and storage requirements
  - Intractability of imperfect state information problems
- Curse of modeling: if "system stochastics" are complex, it is difficult to obtain expressions for the transition probabilities
- Curse of time
  - The data of the problem to be solved is given with little advance notice
  - The problem data may change as the system is controlled—need for on-line replanning

# Next time